Finite Element Method

1 Differential and Variational Formulations

Mathematical representation of physical laws is possible in two alternative but equivalent formulations

- Differential formulation
- Variational formulation

The differential form is usually obtained by first applying the conservation principle on a small box of material followed by a limiting process in which the volume of the box is made infinitesimally small. The variational formulation is the integrated form of the conservation principle over the entire volume of the material under consideration. Each formulation can be obtained from the other one by a relatively straightforward process.

The importance of the variational formulations is based on the following *Characterization Theorem*:

Let V be a linear space, $a(v,v) : V \times V \rightarrow R$ a symmetric positive bilinear form and $f : V \rightarrow R$ a linear functional. Then, the quantity

$$I(v) = \frac{1}{2}a(v,v) - (f,v)$$

attains a minimum over V at u if and only if

$$a(u,v) = (f,v)$$

for all $v \in V$; there is at most one solution of the above.

The problem of finding the function u which makes $I(u) \leq I(v)$ for any $v \in V$ is known as the *functional minimization problem* while that of determining the function u such that $a(u,v) = (f,v)$ for any $v \in V$ is known as the *variational problem*. The Galerkin method is an approximate method of solution of the variational problem.

Using the *calculus of variations* one can show that the function u which solves the above two problems is the solution of a particular differential equation (Euler's equation) with the form

$$Lu = f$$

where L is a linear differential operator.

As a specific example consider the problem of finding the function $u(x)$ satisfying the differential equation

$$-\frac{d^2u}{dx^2} = f(x)$$

subject to the boundary conditions

$$u(0) = 0$$
$$u(1) = 0$$

This is Euler's equation associated with the problem of determining the function $u(x)$ which minimizes the functional

$$I(v) = \frac{1}{2}a(u,v) - (f,v) = \frac{1}{2}(u',v') - (f,v)$$

where $a(u,v) = (u',v') = \int_0^1 u'v'dx$ and $(f,v) = \int_0^1 fvdx$.

Furthermore, the same function $u(x)$ which minimizes the functional (and solves the original differential equation subject to the stated boundary conditions) is also the solution of the variational problem

$$(u',v') = (f,v)$$

2 The Galerkin Method

In the original Galerkin method, one seeks directly for an approximate solution of $a(u,v) = (f,v)$.

One starts by introducing a set of N basis functions $\{\phi_j(x)\}_{j=1}^N$ the domain of which is the entire set of values of the independent variable and that satisfy the stated boundary conditions.

The approximate solution is then simply expressed as

$$u_N = \sum_{j=1}^{N} a_j\phi_j \approx u$$

where u_N is called the *trial function*.

Furthermore, *test functions* are selected so as to be identical to the basis functions, i.e. $\{v_i\}_{i=1}^N = \{\phi_i(x)\}_{i=1}^N$.

The resulting discrete form of the problem is then obtained by substituting the above into the variational formulation yielding

$$\sum_{i=1}^{N} a_j a(\phi_j,\phi_i) = (f,\phi_i)$$

for all $i = 1, 2, ..., N$. Here

$$a(\phi_j, \phi_i) = \int_0^1 \frac{d\phi_j}{dx}\frac{d\phi_i}{dx}dx$$

and

$$(f, \phi_i) = \int_0^1 f\phi_i dx$$

for $i = 1, 2, ..., N$

Using matrix notation, the above is simply written as

$$\mathbf{Ka} = \mathbf{F}$$

Solving the algebraic problem yields the values of the coefficients $a_i, i = 1, 2, ..., N$ of the basis functions in the trial function.

3 The Galerkin Finite Element Method

The Galerkin finite element method is a direct implementation of the original Galerkin procedure in which the chosen basis functions are finite element basis functions (also called sometimes finite element shape functions). These functions are of very simple form (piecewise polynomials of low order are most common) and their most distinctive feature is that they are nonzero only in a small subregion of the computational domain, i.e. the finite element and thus they have a local character, in contrast with the domain-wide basis functions used in the original procedure.

The local finite element basis functions are used to construct global finite element basis functions which are non-zero only on next neighbor elements and the resulting system of algebraic equations can be constructed and assembled together by the computer just as it is created. This feature makes it a trivial process to determine increasingly better approximations by refinement of the initial subdivision. Moreover, the matrix associated with the resulting system has a sparse structure and efficient solution methods are applicable.

In the finite element method one starts by using simple interpolation functions as basis functions at the element level and then proceeds to represent the solution in the entire domain by collecting the contributions associated with each element. The collection of finite elements constitutes the domain.

The basis functions involved at the element level are called local basis functions while the ones that produce the solution in the entire domain are called global basis functions. The role of the local basis functions is to generate the value of the approximated quantity inside the element from the values at the nodes by interpolation while the role of the global functions is to generate the approximate solution in the entire domain. The two sets of basis functions are closely related.

A finite element is thus a triple of entities (K, P_K, Σ) where

- K is a simple geometrical object, such as a line segment in one dimension, a triangle in two dimensions and a parallelepiped in three dimensions,
- P_K is a finite dimensional linear space of basis functions defined on K, and
- Σ is a set of degrees of freedom.

A finite element is then the totality of ingredients associated with the definition of the global approximating function inside the subdomain. Since the method produces approximations to global functions from functions having only local support, it is key to be able to distinguish and to relate the global and local descriptions. Therefore, one needs to carefully specify both local and global parameters as well as their relationship.

Assuming again a one dimensional system, where the domain is subdivided into finite elements and consider an arbitrary interior element e_i where the index i denotes the global node number.

If the values of the required solution at x_i and x_{i+1} are $u(x_i) = u_i$ and $u(x_{i+1}) = u_{i+1}$, respectively, approximate values of $u(x)$, $u(x)^{e_i}$ inside the element can calculated by simple linear interpolation as follows

$$u^{e_i}(x) = u_i\phi_1^i + u_{i+1}\phi_2^i$$

Here, the local finite element basis functions for element e_i, ϕ_1^i and ϕ_2^i are defined as

$$\phi_1^i = \frac{x_{i+1} - x}{h_i}$$

$$\phi_2^i = \frac{x - x_i}{h_i}$$

where $h_i = x_{i+1} - x_i$ is the element size and all the positions are measured in the global coordinate system. Note that the indices 1 and 2 on the local basis functions refer to the local node numbers for the element.

There is a simple relationship between local and global finite element basis functions, specifically, in element e_i,

$$\phi_i = \begin{cases} \phi_2^{i-1} & \text{if } x \in e_{i-1} \\ \phi_1^i & \text{if } x \in e_i \end{cases}$$

For simplicity, assume all finite elements are of equal size (uniformly spaced mesh) so that $h_1 = h_2 = ... = h_i = h$. Now, the approximate solution on the entire domain $u_h(x)$ is represented as a linear combination of the global finite element basis functions, $\phi_i, i = 1, 2, ..., N$, where N is the number of nodes, i.e.

$$u_h(x) = u_1\phi_1 + u_2\phi_2 + ... + u_i\phi_i + u_N\phi_N = \sum_{i=1}^{N} u_i\phi_i$$

Note that in this expression, the u_i's are actually the values of the approximation at the nodal locations.

As in the original Galerkin approach, in the Galerkin finite element method, the finite element trial and test functions are introduced into the variational formulation in order to obtain a discretized version of the problem The final Galerkin finite element method equations can thus be expressed as

$$\sum_{i=1}^{N} u_j a(\phi_j, \phi_i) = (f, \phi_i)$$

for all $i = 1, 2, ..., N$. Here

$$a(\phi_j, \phi_i) = \int_0^1 \frac{d\phi_j}{dx}\frac{d\phi_i}{dx} dx$$

and

$$(f, \phi_i) = \int_0^1 f\phi_i dx$$

for $i = 1, 2, ..., N$.

Using matrix notation, the above is simply written as

$$\mathbf{Ku} = \mathbf{F}$$

Solution of the system of linear algebraic equations yields the values of the approximations u_i at the nodal locations of the finite element mesh.

4 The Finite Element Method for Convection-Diffusion in 1D

Consider the following one-dimensional convection-diffusion equation for the transported quantity $\phi(x)$

$$\rho u \frac{d\phi}{dx} = \Gamma \frac{d^2\phi}{dx^2}$$

for $x \in [0, L]$ subject to suitable boundary conditions such as

$$\phi(0) = 1$$
$$\phi(L) = 1$$

where ρ is the density of the fluid, $u > 0$ is the velocity and Γ the diffusivity of ϕ in the fluid, and all are assumed constant.

Using the finite element method on an uniform mesh (spacing Δx), with piecewise linear global basis functions at the nodes of the finite element mesh yields the following discrete analogue

$$\rho u \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x} = \Gamma(\frac{\phi_{i-1} - 2\phi_i + \phi_{i+1}}{(\Delta x)^2})$$

which is identical to the expression obtained using central differencing in the FDM.

Upwinding must also be used in general to avoid non-physical oscillations.