# Finite Volume Method

## 1 Introduction

An alternative discretization method is based on the idea of regarding the computation domain as subdivided into a collection of finite volumes. In this view, each finite volume is represented by a line segment in 1D, an area in 2D and a volume in 3D. A node, located inside each finite volume is the locus of computational values although values of selected field variables at finite volume boundaries play crucial roles.

In rectangular Cartesian coordinates in 2D the simplest finite volumes are rectangles. For instance, for each node, the rectangle faces are formed by drawing perpendiculars through the midpoints between contiguous nodes. Discretization equations are obtained by integrating the original partial differential equation over the span of each finite volume. The method is easily extended to nonlinear problems.

## 2 The Finite Volume Method in 1D

Consider for instance the following problem consisting of determining the function $\phi(x)$ in $x \in [0, L]$, such that

$$\frac{d^2\phi}{dx^2} + f(x) = 0$$

for $x \in [0, 1]$, where $f(x)$ is a known function and $\phi$ is subject to specified conditions at the boundary such as

$$\phi(0) = 0$$
$$\phi(1) = 0$$

The problem may represent steady state heat conduction in a medium of unit thermal conductivity where energy is being internally generated at the rate $f$, per unit volume and whose boundaries are maintained at zero temperature.

In the finite volume method, to obtain the discrete analogue of the problem, the space domain is first divided into a collection of contiguous finite volumes of size $\Delta x_i$ along the

$x-$direction and unit dimensions along the $y-$ and $z-$ directions. For uniformly sized volumes there is only one spacing $\Delta x$. Any given volume in the set is characterized by the nodal location of its center point $P$ and that of its bounding faces $e$ and $w$ to the East and West of $P$, respectively. Note, that in this arrangement, face $e$ is located half-way between node $P$ and node $E$ pertaining to the finite volume located to its east. Moreover, face $w$ is located half-way between node $P$ and node $W$ pertaining to the finite volume located to its west

Then, the governing equation is then integrated over the span of any given arbitrary volume in the set to yield

$$\frac{d\phi}{dx}|_e - \frac{d\phi}{dx}|_w = \bar{f}(x)\Delta x$$

where $\bar{f}(x)$ is the average value of $f(x)$ inside the finite volume. Note that the derivatives on the left hand side are evaluated at the finite volume faces which are also the mid-point locations between contiguous finite volumes.

The discretized equation is now obtained by approximating the derivatives on the left hand side with central differences, i.e.

$$\frac{\phi_E - \phi_P}{\Delta x} - \frac{\phi_P - \phi_W}{\Delta x} = \bar{f}(x)\Delta x$$

Upon rearrangement this gives

$$-a_W\phi_W + a_P\phi_P - a_E\phi_E = \bar{f}(x)\Delta x$$

where $a_W = a_E = \frac{1}{\Delta x}$, and $a_P = a_W + a_E$.

Note that if $\bar{f}(x) = f(x_P)$ (i.e. the average value of the source term is replaced by the value of $f$ at the nodal location), the above reduces to the simple form

$$\phi_P = \frac{1}{2}[\phi_W + \phi_E + f(x_P)(\Delta x)^2]$$

which is identical to the expression obtained before using the method of finite differences.

# 3 Boundary Conditions

As in the case of finite differences, care is required to properly deal with the boundary conditions at the boundary nodes, specially when derivative conditions are involved.

# 4 Solution of the Algebraic Equations

As was the case with the finite difference method, the resulting system of coupled linear algebraic equations for all nodes in the mesh may sometimes be solved by a direct method (Gauss elimination) but more commonly in CFD, an iterative method is used, most frequently a combination of TDMA with an iterative relaxation-type method.

# 5 The Convection-Diffusion Equation and Upwinding

Consider the following one-dimensional convection-diffusion equation for the transported quantity $\phi(x)$

$$\rho u \frac{d\phi}{dx} = \Gamma \frac{d^2\phi}{dx^2}$$

for $x \in [0, L]$ subject to suitable boundary conditions such as

$$\phi(0) = 1$$
$$\phi(L) = 1$$

where $\rho$ is the density of the fluid, $u > 0$ is the velocity and $\Gamma$ the diffusivity of $\phi$ in the fluid, and all are assumed constant.

Using the finite volume method on an uniform mesh (spacing $\Delta x$), and integrating once over the span of the volume centered at $x_P$ with faces $e$ and $w$, as described above yields

$$(\rho u \phi)|_e - (\rho u \phi)|_w = (\Gamma \frac{d\phi}{dx})|_e - (\Gamma \frac{d\phi}{dx})|_w$$

Evaluating $\phi_e$ and $\phi_w$ by linear interpolation and using central differencing for the derivatives on the right hand side yields

$$\frac{1}{2}(\rho u)(\phi_E + \phi_P) - \frac{1}{2}(\rho u)(\phi_P + \phi_W) = \Gamma(\frac{\phi_E - \phi_P}{\Delta x}) - \Gamma(\frac{\phi_E - \phi_P}{\Delta x})$$

Rearranging gives

$$-a_E \phi_E + a_P \phi_P - a_W \phi_W = 0$$

where

$$a_E = \frac{\Gamma}{\Delta x} - \frac{1}{2}(\rho u)$$
$$a_W = \frac{\Gamma}{\Delta x} + \frac{1}{2}(\rho u)$$

and

$$a_P = a_E + a_W$$

As with the FD method, the above scheme works fine as long as the Peclet number of the system, defined as

$$Pe = \frac{\rho u L}{\Gamma}$$

and representing the ratio of convective to diffusive transports, is small (say $< 1$). Upwinding must be used in general to avoid non-physical oscillations. In the case of the finite volume method, upwinding is implemented as follows:

If $u_e > 0$, then $\phi_e = \phi_P$, and if $u_e < 0$, then $\phi_e = \phi_E$. Moreover, if $u_w > 0$, then $\phi_w = \phi_W$, and if $u_w < 0$, then $\phi_w = \phi_P$. As with the FDM, this scheme is only first order accurate and it has the effect of introducing a non-physical diffusive effect in the solution called false diffusion. Discretization schemes of higher order of accuracy have been developed based on the notion of upwinding (e.g. the exponential scheme).